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A Weierstrass Theorem for a Complex Separable Hilbert Space*

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In a recent paper, Prenter [3] showed that if H is a real, separable Hilbert space then the family of all continuous polynomials from H to H is dense in the space of continuous functions on a compact subspace of H. At the end of her paper she conjectures the validity of a Weierstrass approximation theorem for complex Hilbert spaces. We found an alternative version of such a theorem to which we were led by symmetry considerations.

The proof of our theorem is in two parts. The first treats the finite dimensional case. The second (which we omit) extends the result to the infinite dimensional case by projecting onto a finite dimensional subspace and then using some simple Hilbert space inequalities. This part follows [3 Sect. 5] and is equally valid in the real or complex case.

We now proceed with the theorem in the finite dimensional case.

THEOREM. *Let H be a complex separable Hilbert space. The family of continuous polynomials on H restricted to a compact subset K of H is dense in the set $C(K)$ of continuous functions on H into H restricted to K where $C(K)$ carries the uniform norm topology.*

Proof. As mentioned our space H is finite dimensional. Let $\varphi_1 \;\varphi_2 \cdots \varphi_m$ be a fixed orthonormal basis. If $x \in H$, we have $x = \sum_{i=1}^{m} z_i \varphi_i$. Define $\bar{x} = \sum_{i=1}^{m} \bar{z}_i \varphi_i$. For p and q, any two non-negative integers, define

$$x^{p,q} = (\underbrace{x, x,\ldots, x}_{p\text{-times}}, \underbrace{\bar{x}, \bar{x},\ldots, \bar{x}}_{q\text{-times}}) \in H^{p+q},$$

where H^k is the cartesian product of H with itself k-times. For convenience, we set $x^{0,0}$ equal to the zero vector.

A k-linear operator L_k is a function from H^k into H which is linear in each variable. (For $k = 0$, L_0 is a constant function from H into H).

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For each n ($n = 0, 1, 2, \ldots$) suppose we are given $n + 1$ n-linear operators $L_n^0, L_n^1, \ldots, L_n^n$.

A sum of the form

$$\sum_{n=0}^{N} \sum_{k=0}^{n} L_n^k(x^{k,n-k})$$

will be called an Nth degree polynomial operator. For a given n and k ($n = 1, 2, 3, \ldots, 0 \leqslant k \leqslant n$) we compute $L_n^k(x^{k,n-k})$ where $x = \sum_{i=1}^{m} z_i \varphi_i$.

$$\begin{aligned} L_n^k(x^{k,n-k}) &= L_n^k(x, x, \ldots, x, \bar{x}, \bar{x}, \ldots, \bar{x}) \\ &= L_n^k\left(\sum_{i=1}^{m} z_i\varphi_i, \ldots, \sum_{i=1}^{m} z_i\varphi_i, \sum_{i=1}^{m} \bar{z}_i\varphi_i, \ldots, \sum_{i=1}^{m} \bar{z}_i\varphi_i\right) \\ &= \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_k=1}^{m} \sum_{j_1=1}^{m} \cdots \sum_{j_{n-k}=1}^{m} z_{i_1} z_{i_2} \cdots z_{i_k} \bar{z}_{j_1} \bar{z}_{j_2} \cdots \bar{z}_{j_{n-k}} \\ &\quad \cdot L_n^k(\varphi_{i_1}, \varphi_{i_2}, \ldots, \varphi_{i_k}, \varphi_{j_1}, \varphi_{j_2}, \ldots, \varphi_{j_{n-k}}). \end{aligned}$$

Note that the coefficient of

$$L_n^k(\varphi_{i_1}, \varphi_{i_2}, \ldots, \varphi_{i_k}, \varphi_{j_1}, \varphi_{j_2}, \ldots, \varphi_{j_{n-k}}),$$

that is, $z_{i_1} z_{i_2} \cdots z_{i_k} \bar{z}_{j_1} \bar{z}_{j_2} \cdots \bar{z}_{j_{n-k}}$ contains exactly k z's and $(n - k)$ $\bar{z}$'s.

Let K be a compact subset of H. We will use the following isometry

$$\psi \colon H \to l^2(m)$$

where $x = \sum_{i=1}^{m} z_i\varphi_i$ and $\psi(x) = (z_1, z_2, \ldots, z_m)$.

Let F be a continuous function from H into H. Since $F(x)$ is in H we have

$$F(x) = \sum_{i=1}^{m} f_i\varphi_i = \sum_{i=1}^{m} f_i(\psi(x))\, \varphi_i = \sum_{i=1}^{m} f_i(z_1, z_2, \ldots, z_m)\, \varphi_i.$$

It is clear that each $f_i (1 \leqslant i \leqslant m)$ is a continuous function of m complex variables on the set $\psi(H) = l^2(m)$.

Since K is compact, and ψ is continuous, $\psi(K)$ is also compact. Let $\epsilon > 0$. By the Weierstrass approximation theorem for m complex variables [2], there are polynomials $p_i = p_i(z_1, z_2, \ldots, z_m) (1 \leqslant i \leqslant m)$ defined on $\psi(H)$ such that

$$\| f_i - p_i \| = \sup_{x\in K} | f_i(\psi(x)) - p_i(\psi(x))| \leqslant \epsilon/(m)^{1/2}.$$

Let $P(x) = \sum_{i=1}^{m} p_i\varphi_i = \sum_{i=1}^{m} p_i(\psi(x))\, \varphi_i$.

Using the appropriate norms we have

$$\|F - P\|^2 = \left\| \sum_{i=1}^{m} f_i \varphi_i - \sum_{i=1}^{m} p_i \varphi_i \right\|^2$$

$$= \left\| \sum_{i=1}^{m} (f_i - p_i)\, \varphi_i \right\|^2 = \sum_{i=1}^{m} \|(f_i - p_i)\, \varphi_i\|^2$$

$$= \sum_{i=1}^{m} \|f_i - p_i\|^2 \leqslant \sum_{i=1}^{m} \frac{\epsilon^2}{m} = \epsilon^2.$$

So $\|F - P\| \leqslant \epsilon$.

We have only to show that P can be written in the form

$$\sum_{n=0}^{N} \sum_{k=0}^{n} L_n{}^k(x^{k,n-k})$$

for some N and some choice of the $L_n{}^k$'s, i.e., that P is really an Nth degree polynomial operator.

Let d_i equal the degree of p_i and $N \geqslant \max_{1 \leqslant i \leqslant m} d_i$. This is the desired N. Consider $p_i \varphi_i$ $(1 \leqslant i \leqslant m)$. Since p_i is a polynomial, $p_i \varphi_i$ will be a sum consisting of each term of p_i multiplied by φ_i. So we may interpret $p_i \varphi_i$ as a polynomial with vector coefficients. Thus, $P = \sum_{i=1}^{m} p_i \varphi_i$ is also a polynomial, of degree $\leqslant N$, with vector coefficients. Consider an arbitrary term in P,

$$\tilde{v} z_{i_1}^{\lambda_1} z_{i_2}^{\lambda_2} \cdots z_{i_r}^{\lambda_r} \bar{z}_{j_1}^{\mu_1} \bar{z}_{j_2}^{\mu_2} \cdots \bar{z}_{j_s}^{\mu_s}$$

where $1 \leqslant \lambda_i (1 \leqslant i \leqslant r)$, $1 \leqslant \mu_j (1 \leqslant j \leqslant s)$, and where we now specify that $i_1 < i_2 < \cdots < i_r$, $j_1 < j_2 < \cdots < j_s$. Setting $u = \lambda_1 + \lambda_2 + \cdots + \lambda_r$ and $v = \mu_1 + \mu_2 + \cdots + \mu_s$, it follows that the degree of the above term is $u + v \leqslant$ degree $P \leqslant N$ and $\tilde{v}$ is its vector coefficient. We will also consider $\sum_{n=0}^{N} \sum_{k=0}^{n} L_n{}^k(x^{k,n-k})$ as a polynomial of degree $\leqslant N$ with vector coefficients.

The only place where terms of degree $u + v$ occur is in

$$\sum_{k=0}^{u+v} L_{u+v}^{k}(x^{k,u+v-k}).$$

In this sum, the only place where a term of degree u in the z_i's and of degree v in the $\bar{z}_j$'s occurs is in $L_{u+v}^{u}(x^{u,v})$. A given term might occur more than once in $L_{u+v}^{u}(x^{u,v})$.

We now define the $L_n{}^k$'s on certain sets of points. Consider

$$L_n{}^k(\varphi_{i_1}, \varphi_{i_2}, \dots, \varphi_{i_k}, \varphi_{j_1}, \varphi_{j_2}, \dots, \varphi_{j_{n-k}}). \tag{1}$$

If the following two conditions are not satisfied, define (1) to be the zero vector:

(i) $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k$

(ii) $j_1 \leqslant j_2 \leqslant \cdots \leqslant j_{n-k}$.

If the above two conditions are satisfied we define the $L_n{}^k$'s below. In $L_{u+v}^u(x^{u,v})$, the term

$$z_{i_1}^{\lambda_1} z_{i_2}^{\lambda_2} \cdots z_{i_r}^{\lambda_r} \bar{z}_{j_1}^{\mu_1} \bar{z}_{j_2}^{\mu_2} \cdots \bar{z}_{j_s}^{\mu_s} \tag{2}$$

occurs exactly once, with a vector coefficient of

$$L_{u+v}^u(\varphi_{i_1}, \varphi_{i_1}, \dots, \varphi_{i_1}, \varphi_{i_2}, \varphi_{i_2}, \dots, \varphi_{i_2}, \dots, \varphi_{i_r}, \varphi_{i_r}, \dots, \varphi_{i_r}, \varphi_{j_1}, \varphi_{j_1}, \dots, \varphi_{j_1},$$
$$\varphi_{j_2}, \varphi_{j_2}, \dots, \varphi_{j_2}, \dots, \varphi_{j_s}, \varphi_{j_s}, \dots, \varphi_{j_s}), \tag{3}$$

where i_k occurs λ_k times ($1 \leqslant k \leqslant r$) and j_l occurs μ_l times ($1 \leqslant l \leqslant s$).

Now we define (3) $= \tilde{v}$. That is, we set the vector coefficient of (2) in $L_{u+v}^u(x^{u,v})$ equal to the vector coefficient of (2) in P. Now each term in P occurs with the proper vector coefficient in $\sum_{n=0}^N \sum_{k=0}^n L_n{}^k(x^{k,n-k})$. For all those points not yet considered set the $L_n{}^k$'s in $\sum_{n=0}^N \sum_{k=0}^n L_n{}^k(x^{k,n-k})$ equal to the zero vector. We then have $P(x) = \sum_{n=0}^N \sum_{k=0}^n L_n{}^k(x^{k,n-k})$, since, if we view each as a polynomial, like terms have the same vector coefficient.

We now extend each $L_n{}^k$ linearly in each variable to obtain k-linear operators. This shows that P is of the desired form and so completes the proof.

In the above we considered $n+1$ n-linear operators $L_n{}^0, L_n{}^1, \dots, L_n{}^n$ $(n = 1, 2, \dots)$ and using these formed the sums $\sum_{n=0}^N \sum_{k=0}^n L_n{}^k(x^{k,n-k})$. Prenter only uses one n-linear operator and looks at sums of the form $\sum_{n=0}^N \sum_{k=0}^n L_n(x^{k,n-k})$. These sums have certain symmetry properties and therefore cannot be used to approximate certain unsymmetric functions.

Consider for example $z_1{}^2$. This term will occur in $\sum_{n=0}^N \sum_{k=0}^n L_n(x^{k,n-k})$ only in $\sum_{k=0}^2 L_2(x^{k,2-k}) = L_2(x, x) + L_2(x, \bar{x}) + L_2(\bar{x}, \bar{x})$. In fact, $z_1{}^2$ will occur only in $L_2(x, x) = \sum_{i=1}^m \sum_{j=1}^m z_i z_j L_2(\varphi_i, \varphi_j)$. And then only in the form $z_1 z_1 L_2(\varphi_1, \varphi_1)$. Similarly $\bar{z}_1{}^2$ will occur only in $\sum_{k=0}^2 L_2(x^{k,2-k})$. In fact, $\bar{z}_1{}^2$ will only occur in $L_2(\bar{x}, \bar{x}) = \sum_{i=1}^m \sum_{j=1}^m \bar{z}_i \bar{z}_j L_2(\varphi_i, \varphi_j)$ and then only in the form $\bar{z}_1 \bar{z}_1 L_2(\varphi_1, \varphi_1)$. So $z_1{}^2$ and $\bar{z}_1{}^2$ both have the same vector coefficient $L_2(\varphi_1, \varphi_1)$. This is the symmetry consideration. Other examples are of course possible.

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